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On von Neumann regular rings with weak comparability

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Abstract

In 1991, K.C. O'Meara first defined the notion of weak comparability for regular rings, and he showed that every simple directly finite regular ring with weak comparability is unit-regular. In this paper, we investigate properties for regular rings with weak comparability, and we show that the strict cancellation property and the strict unperforation property hold for the family of directly finite finitely generated projective modules over these rings.

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Introduction

The notion of weak comparability was first introduced by O'Meara [16], to prove that simple directly finite regular rings with weak comparability must be unit-regular [7, Open Problem 3]. Thereafter properties for simple regular rings with weak comparability have been studied in other papers ([1,3,4] etc.), and as an interesting result, it was shown in [3] that a simple regular ring R satisfies weak comparability if and only if the strict unperforation property holds for the family of finitely generated projective R -modules, i.e., $nA < nB$ implies $A < B$ for any positive integer n and any finitely generated projective R -modules A and B . Recently unit-regular rings with weak comparability also have been studied in [13,15]. In this paper, we investigate the properties for regular rings with weak comparability. In Section 1, we show that directly finite regular rings with weak comparability are stably finite (Corollary 1.6(2)) and regular rings R with weak comparability satisfy the

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property special (DF), i.e., nP is directly finite for any positive integer n and any directly finite projective R -module P (Theorem 1.8). In Section 2, for a regular ring R with weak comparability, we also prove the following theorems by using results in Section 1:

- (1) If $A \oplus C \prec B \oplus C$ for some finitely generated projective R -modules A , B and C such that C is directly finite, then $A \prec B$ (Theorem 2.2).
- (2) For $M \in \text{SubP}(R)$, if M contains no infinite direct sums of nonzero pairwise isomorphic submodules, then so does nM for any positive integer n , where $\text{SubP}(R)$ means the family of all R -modules N such that N is subisomorphic to some projective R -module (Theorem 2.6).
- (3) If $nA \prec nB$ for some positive integer n and some finitely generated projective R -modules A and B such that B is directly finite, then $A \prec B$ (Theorem 2.10).

We note that (2) was treated as a generalization for [7, Open Problem 9] in case of regular rings with weak comparability, and we can refer to [13, Theorem 2.6] for other regular rings.

For the proofs of results in Section 2, the property special (DF) works usefully, and hence we give some backgrounds to the properties (DF) and special (DF), where a ring R satisfies (DF) if $P \oplus Q$ is directly finite for any directly finite projective R -modules P and Q . The notion of (DF) was born from the study of directly finite projective modules over directly finite regular rings with the comparability axiom [9], and the definition for it was first given in [10]. In [11] we showed that unit-regular rings with s -comparability have (DF), and using this result effectively, we could determine the form of directly finite projective modules over these rings. After that, more generally it was shown that regular rings with s -comparability have (DF) in [12], and also Ara, Pardo, and Perera [5] dealt with (DF) as a way of characterizing directly finiteness of idempotents in multiplier rings. But the property (DF) can fail even for regular rings with general comparability (see [10]), and so we treated the problem “Does every regular ring have special (DF)?” It is unknown that there exists a counter-example for it. For positive answers, it is known that the following typical regular rings have special (DF): unit-regular rings with weak comparability, regular rings with general comparability, right \aleph_0 -continuous regular rings, N^* -complete regular rings, regular rings whose primitive factor rings are artinian, etc. (see [13]). We notice that the above problem is a generalization of [7, Open Problem 1] for projective modules over regular rings, and that the properties (DF) and special (DF) play a major role not only in studying properties of directly finite projective modules over regular rings but also in studying cancellation properties of idempotents in multiplier rings of regular rings.

Throughout this paper, R is a ring with identity and R -modules are unitary right R -modules. We begin with some notations and definitions.

Notation

Given R -modules P and Q , we use $P \lesssim Q$ (respectively $P \lesssim_{\oplus} Q$, $P \prec Q$, $P \prec_{\oplus} Q$) to mean that there exists an isomorphism from P to a submodule of Q (respectively a direct summand of Q , a proper submodule of Q , a proper direct summand of Q). For a submodule P of an R -module Q , $P \leqslant_{\oplus} Q$ (respectively $P < Q$, $P <_{\oplus} Q$) means that P is a direct

summand of Q (respectively a proper submodule of Q , a proper direct summand of Q). For a cardinal number k and an R -module P , kP denotes the direct sum of k -copies of P . We denote the socle of an R -module M by $\text{Soc}(M)$.

Definition. An R -module P is *directly finite* provided that P is not isomorphic to a proper direct summand of itself. If P is not directly finite, then P is said to be *directly infinite*. Note that every direct summand of a directly finite module is directly finite, and every directly infinite module has an infinite direct sum of nonzero pairwise isomorphic submodules [7, Corollary 5.6]. A ring R is *directly finite* (respectively *directly infinite*) if the R -module R is directly finite (respectively directly infinite), and R is said to be *stably finite* if the ring $M_n(R)$ of $n \times n$ matrices over R is directly finite for all positive integers n . It is well known from [7, Lemma 5.1] that an R -module M is directly finite if and only if so is $\text{End}_R(M)$, and hence a ring R is stably finite if and only if every finitely generated projective R -module is directly finite. A ring R is said to be *regular* if for each $x \in R$, there exists an element y of R such that $xyx = x$, and R is said to be *unit-regular* if for each $x \in R$, there exists a unit element (i.e., an invertible element) u of R such that $xux = x$.

We recall some properties satisfied by regular rings: Let R be a regular ring, and let P be a projective R -module. Then

- (a) every finitely generated submodule of P is a direct summand of P [7, Theorem 1.11];
- (b) P is a direct sum of cyclic submodules, each of which is isomorphic to a principal right ideal of R .

Definition. An R -module M has *the exchange property* if for every R -module A and any decompositions $A = M' \oplus N = \bigoplus_{i \in I} A_i$ with $M' \cong M$, there exist submodules $A'_i \leq A_i$ such that $A = M' \oplus (\bigoplus_{i \in I} A'_i)$. It follows from the modular law that the above A'_i must be a direct summand of A_i for each i . If the above condition is satisfied whenever the index set I is finite, M is said to satisfy *the finite exchange property*. For these properties, Oshiro [17] showed that every projective module over a regular ring satisfies the finite exchange property, and more generally, it was proved that every projective module over a regular ring satisfies the exchange property in [9, Lemma 2], which was given to the author by K. Oshiro. This result is frequently used in this paper.

All basic results concerning regular rings can be found in K.R. Goodearl's book [7].

1. Directly finite projective modules

We recall the definition of weak comparability for regular rings.

Definition [16]. A regular ring R satisfies *weak comparability* if for each nonzero $x \in R$, there exists a positive integer $n = n(x)$ such that $n(yR) \lesssim R$ implies $yR \lesssim xR$ for all $y \in R$.

Lemma 1.1. *Let R be a regular ring with weak comparability, and let P be a cyclic projective R -module. Then the following conditions are equivalent:*

- (a) P is directly finite.
- (b) For each nonzero R -module X , there exists a positive integer n such that $nX \not\lesssim P$.
- (c) For any nonzero R -module X , we have that $\aleph_0 X \not\lesssim P$.

Proof. (b) \Rightarrow (c) \Rightarrow (a) are obvious.

(a) \Rightarrow (b). Assume that P is directly finite and there exists a nonzero cyclic R -module X such that $nX \lesssim P$ for all positive integers n . Since P is cyclic projective, we have $P \lesssim_{\oplus} R$. X is a cyclic module, so that $2X$ is finitely generated. Since $2X \lesssim P \lesssim_{\oplus} R$, we have $2X \lesssim_{\oplus} R$ by [7, Theorem 1.11]. In fact, this argument shows that $(2m)X \lesssim_{\oplus} R$ for all $m \geq 1$. By the weak comparability of R , there exists a positive integer $k = n(X)$ such that $kY \lesssim R$ implies $Y \lesssim X$ for all cyclic projective R -module Y , and so $2X \lesssim_{\oplus} X$. Since P is directly finite and $X \lesssim_{\oplus} P$, we see that X is directly finite and so $X = 0$, which is a contradiction. Therefore (a) \Rightarrow (b) holds. \square

Lemma 1.2. *Let R be a regular ring. Let $A = \bigoplus_{i \in I} A_i$ be a projective R -module, and let $\{X_j\}_{j=1}^{\infty}$ be an independent family of finitely generated submodules of A . Then there exist an independent family $\{A_{ij}\}_{j=1}^{\infty}$ of submodules of A_i for each $i \in I$ such that $X_j \cong \bigoplus_{i \in I} A_{ij}$ for each $j = 1, 2, \dots$*

Proof. Since X_1 is a finitely generated submodule of A , we see that $X_1 \leq_{\oplus} A = \bigoplus_{i \in I} A_i$. Using the exchange property for projective modules over regular rings, we have decompositions $A_i = A_{i1} \oplus A_{i1}^*$ for each $i \in I$ such that $A = X_1 \oplus (\bigoplus_{i \in I} A_{i1}^*)$, from which $X_1 \cong \bigoplus_{i \in I} A_{i1}$. Noting that $X_1 \oplus X_2 \leq_{\oplus} A = X_1 \oplus (\bigoplus_{i \in I} A_{i1}^*)$, we have decompositions $A_{i1}^* = A_{i2} \oplus A_{i2}^*$ for each $i \in I$ such that $A = X_1 \oplus X_2 \oplus (\bigoplus_{i \in I} A_{i2}^*)$ and $X_2 \cong \bigoplus_{i \in I} A_{i2}$. Continuing the above procedure, we have a desired one. \square

Proposition 1.3. *Let R be a regular ring with weak comparability. Let P and Q be projective R -modules such that P is cyclic directly finite. If $\aleph_0 X \lesssim P \oplus Q$ for some nonzero cyclic R -module X , then $\aleph_0 X \lesssim Q$.*

Proof. Since X is cyclic and subisomorphic to a projective module by hypothesis, we see that X is cyclic projective. Note that $\aleph_0 X \lesssim P \oplus Q$, from which there exists an independent family $\{X_i\}_{i=1}^{\infty}$ of nonzero submodules of $P \oplus Q$ such that $X_i \cong X$. By Lemma 1.2, there exist independent families $\{Y_i\}_{i=1}^{\infty}$ of submodules of P and $\{Z_i\}_{i=1}^{\infty}$ of submodules of Q such that $Y_i \oplus Z_i \cong X_i$ for each positive integer i . Noting that P is cyclic directly finite projective, we may assume from Lemma 1.1 that $Z_i \neq 0$ for all i . Since $(X \cong) Y_2 \oplus Z_2 \cong Y_3 \oplus Z_3$, we have $Y_3^* \leq_{\oplus} Y_3$ and $Z_3^* \leq_{\oplus} Z_3$ such that $Y_2 \cong Y_3^* \oplus Z_3^*$. Noting that $Y_3 \oplus Z_3 \cong Y_4 \oplus Z_4$, we have $Y_4^* \leq_{\oplus} Y_4$ and $Z_4^* \leq_{\oplus} Z_4$ such that $Y_3^* \cong Y_4^* \oplus Z_4^*$. Continuing the above procedure, we have that $X \cong Y_2 \oplus Z_2 \cong Z_2 \oplus (Z_3^* \oplus Y_3^*) \cong Z_2 \oplus Z_3^* \oplus (Z_4^* \oplus Y_4^*) \cong \dots$. By the weak comparability of R , there exists a positive integer $m = n(Z_1)$ such that $mA \lesssim R$ implies $A \lesssim Z_1$ for all cyclic projective R -modules A . Since $mY_{m+1}^* \lesssim Y_2 \oplus \dots \oplus Y_{m+1} \leq P \lesssim R$ and Y_{m+1}^* is cyclic projective, we have $Y_{m+1}^* \lesssim Z_1$.

Therefore $X \cong Z_2 \oplus Z_3^* \oplus \cdots \oplus Z_{m+1}^* \oplus Y_{m+1}^* \lesssim Z_1 \oplus Z_2 \oplus Z_3^* \oplus \cdots \oplus Z_{m+1}^* \leq \oplus Z_1 \oplus Z_2 \oplus Z_3 \oplus \cdots \oplus Z_{m+1}$ ($\leq Q$). Continuing the procedure above for families $\{Y_i\}_{i=m+2}^\infty$ and $\{Z_i\}_{i=m+2}^\infty$, we can prove that $\aleph_0 X \lesssim Q$ as desired. \square

Theorem 1.4. *Let R be a regular ring with weak comparability.*

- (I) *Let P be a finitely generated projective R -module with a cyclic decomposition $P = \bigoplus_{i=1}^n P_i$. Then the following conditions are equivalent:*
- (a) *P is directly finite.*
 - (b) *P_i is directly finite for each $i = 1, 2, \dots, n$.*
 - (c) *$\aleph_0 X \not\lesssim P$ for all nonzero R -modules X .*
- (II) *Let P be a non-finitely generated projective R -module with a cyclic decomposition $P = \bigoplus_{i \in I} P_i$. Then the following conditions are equivalent:*
- (a) *P is directly finite.*
 - (b) *$\aleph_0 X \not\lesssim P$ for all nonzero R -modules X .*
 - (c) *P_i is directly finite for each $i \in I$, and for each nonzero R -module X there exists a finite subset F of I such that $X \not\lesssim \bigoplus_{i \in I-F} P_i$.*
 - (d) *P_i is directly finite for each $i \in I$, and $\aleph_0 X \not\lesssim_{\oplus} P$ for all nonzero R -modules X .*

Proof. (I) (c) \Rightarrow (a) \Rightarrow (b) are obvious, and (b) \Rightarrow (c) follows from Lemma 1.1 and Proposition 1.3.

(II) (b) \Rightarrow (a) \Rightarrow (d) are obvious, and (c) \Rightarrow (b) follows from Lemma 1.1 and Proposition 1.3 again. We only show that (d) \Rightarrow (c) holds. Assume that each P_i is directly finite and (c) does not hold. Since there exists a nonzero R -module X such that $X \lesssim \bigoplus_{i \in I-F} P_i$ for all finite subsets F of I , we have a nonzero cyclic submodule X' of X such that $X' \lesssim \bigoplus_{i \in I-F} P_i$ for all finite subsets F of I . Then X' is nonzero cyclic projective and $\aleph_0 X' \lesssim_{\oplus} P$, contradicting our hypotheses. Therefore (d) \Rightarrow (c) holds. \square

Remark 1.5. From Theorem 1.4 above, we see that for a projective R -module P over a regular ring with weak comparability, P is directly finite if and only if $\aleph_0 X \not\lesssim P$ for all nonzero R -modules X . We recall that an analogous result was proved in the case of regular rings whose primitive factor rings are artinian (see [14, Theorem 1.3]). But we notice that the above criterion of directly finiteness for projective modules over regular rings is false in general by [7, Example 5.15].

Corollary 1.6. *Let R be a regular ring with weak comparability.*

- (1) *If P and Q are directly finite projective R -modules such that P is finitely generated, then $P \oplus Q$ is directly finite.*
- (2) *If R is directly finite, then R is stably finite.*

Proof. (1) follows from Proposition 1.3 and Remark 1.5. Note that for each positive integer n , $M_n(R)$ is directly finite if and only if so is nR . (2) follows from the above (1). \square

Lemma 1.7 [1, Lemma 3.3]. *Let R be a regular ring, and let P and Q be finitely generated projective R -modules with $P \lesssim nQ$ for some positive integer n . Then there exists a decomposition $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ such that $P_n \lesssim \cdots \lesssim P_1 \lesssim Q$.*

Theorem 1.8. *Let R be a regular ring with weak comparability. Then R has the property special (DF), i.e., nP is directly finite for any directly finite projective R -module P and any positive integer n .*

Proof. We may assume with no loss of generality that $n = 2$, and that P is non-finitely generated projective by Corollary 1.6(1). Assume that P is directly finite and $2P$ is directly infinite. Let $P = \bigoplus_{i \in I} P_i$ be a cyclic decomposition of P , and so each P_i is directly finite. Since $2P$ is directly infinite, using Theorem 1.4(II)(c), we have a nonzero cyclic R -module X such that $X \lesssim 2P_{i_0}$, $X \lesssim 2Y_1$, $X \lesssim 2Y_2, \dots$, where $\{Y_i\}_{i=1}^\infty$ is an independent family of finitely generated submodules of P , and each Y_i is written by a finite direct sum of P_i (where $i \in I - \{i_0\}$). Note that X is nonzero cyclic projective. Since $X \lesssim 2P_{i_0}$, by Lemma 1.7 we have a decomposition $X = X_0 \oplus X_0^*$ such that $X_0^* \lesssim X_0 \lesssim P_{i_0}$, and so $X_0 \neq 0$. On the other hand, noting that $X \lesssim 2Y_1$, we have a decomposition $X = X_1 \oplus X_1^*$ such that $X_1^* \lesssim X_1 \lesssim Y_1$, and so $2X_1^* \lesssim X$. Next, noting that $X_1^* \lesssim X \lesssim 2Y_2$, we have a decomposition $X_1^* = X_2 \oplus X_2^*$ such that $X_2^* \lesssim X_2 \lesssim Y_2$, and so $2^2 X_2^* \lesssim 2X_1^* \lesssim X$. Continuing the above procedure, we have a decomposition $X = X_1 \oplus \cdots \oplus X_n \oplus X_n^*$ such that $2^n X_n^* \lesssim X \lesssim R$. Using the weak comparability of R , there exists a positive integer $m = n(X_0)$ such that $X_m^* \lesssim X_0$. Hence $X = X_1 \oplus \cdots \oplus X_m \oplus X_m^* \lesssim X_0 \oplus X_1 \oplus \cdots \oplus X_m \lesssim P_{i_0} \oplus \cdots \oplus P_{i_k}$ for some k . Repeating the above argument we can prove that $\aleph_0 X \lesssim P$. But this is a contradiction for the direct finiteness of P by Theorem 1.4(II). The proof is complete. \square

Notes 1.9. (1) It is well known that the property (DF) does not hold for regular rings with weak comparability in general by [10, Example] with [16, Proposition 2], and that every simple regular ring R with weak comparability has (DF) if and only if R satisfies 2-comparability by [5, Theorem 4.4] with [3, Theorem 4.3], where a ring R has the property (DF) if $P \oplus Q$ is directly finite for any directly finite projective R -modules P and Q .

(2) In general we see that if a ring R has the property special (DF) (respectively (DF)), then so does a ring S which is Morita equivalent to R . Therefore for a regular ring R with weak comparability, each $M_n(R)$ has special (DF) by Theorem 1.8.

Proposition 1.10. *Let R be a simple directly finite regular ring with weak comparability. Then every directly finite projective R -module is countably generated.*

Proof. Assume that there exists an uncountably generated directly finite projective R -module with a cyclic decomposition $P = \bigoplus_{i \in I} P_i$, where I is an uncountable set. For each positive integer n , set $I_n = \{i \in I \mid R \lesssim nP_i\}$. Noting that R is simple, we see from [7, Corollary 2.23] that $I = \bigcup_{i=1}^\infty I_n$, and so I_n is an uncountable set for some n because I is an uncountable set. Then we have that $\aleph_0 R \lesssim n(\bigoplus_{i \in I_n} P_i)$, from which $n(\bigoplus_{i \in I_n} P_i)$ is directly infinite by Theorem 1.4(II). By Theorem 1.8, we have that $\bigoplus_{i \in I_n} P_i$ is directly infinite.

Noting that $\bigoplus_{i \in I_n} P_i \leq \bigoplus P$, we see that P is directly infinite, which is a contradiction for the assumption. The proof is complete. \square

Proposition 1.11. *Let R be a directly infinite regular ring with weak comparability. Then every directly finite projective R -module is finitely generated. If in addition R is simple, then R has no nonzero directly finite projective R -modules [5, Proof of Theorem 4.4].*

Proof. Let R be a directly infinite regular ring with weak comparability, and let P be a non-finitely generated projective R -module with a nonzero cyclic decomposition $P = \bigoplus_{i \in I} P_i$, where I is a non-finite set. Since R is directly infinite, we have a nonzero cyclic projective R -module X such that $\aleph_0 X \lesssim R$. By the weak comparability of R , we see that $X \lesssim_{\oplus} P_i$ for all $i \in I$, and so $\aleph_0 X \lesssim_{\oplus} P$. Therefore P is directly infinite. In addition we assume that R is simple. We give a different proof from [5, Proof of Theorem 4.4], using Theorem 1.8. We claim that every nonzero cyclic projective R -module is directly infinite. Let X be a nonzero cyclic projective R -module. Then $R \lesssim_{\oplus} nX$ for some positive integer n by the simplicity of R , and so nX is directly infinite. Hence X is directly infinite from Theorem 1.8. Therefore every nonzero projective R -module is directly infinite. The proof is complete. \square

Example 1.12. There exists a simple directly infinite regular ring with weak comparability. Because, let V_F be an infinite-dimensional vector space over a field F , and set $Q = \text{End}_F(V)$ and $M = \{x \in Q \mid \dim_F(xV) < \dim_F(V)\}$. Then Q/M is a purely infinite simple regular ring, and thus is directly infinite and satisfies weak comparability (see [2, Example 1.3 and Corollary 2.2]). \square

2. Strict unperforation

We consider the strict cancellation property for the family of finitely generated projective modules over regular rings with weak comparability. To see this, we need the following lemma.

Lemma 2.1. *Let R be a regular ring, and let $X \oplus C' \oplus Y \leq \bigoplus B \oplus C \oplus Y$ with an isomorphism f from C to C' for some finitely generated projective R -modules B, C and Y . Then there exist decompositions $B = B_1 \oplus B_1^*$ and $B_n^* = B_{n+1} \oplus B_{n+1}^*$ ($n \geq 1$); $C = C_1 \oplus C_1^*$ and $C_n^* = C_{n+1} \oplus C_{n+1}^*$ ($n \geq 1$) such that $X \cong B_1 \oplus C_1$, $C_n \cong B_{n+1} \oplus C_{n+1}$ ($n \geq 1$) and $B \oplus C \oplus Y = X \oplus Y \oplus fC_1 \oplus \cdots \oplus fC_n \oplus B_{n+1}^* \oplus C_{n+1}^*$ for each $n = 0, 1, 2, \dots$, where $C_0 = 0$.*

Proof. Using the exchange property for $X \oplus Y$, there exist decompositions $B = B_1 \oplus B_1^*$ and $C = C_1 \oplus C_1^*$ such that $B \oplus C \oplus Y = X \oplus Y \oplus B_1^* \oplus C_1^*$, and hence $X \cong B_1 \oplus C_1$. Since $X \oplus Y \oplus fC_1 \leq \bigoplus B \oplus C \oplus Y = X \oplus Y \oplus B_1^* \oplus C_1^*$, there exist decompositions $B_1^* = B_2 \oplus B_2^*$ and $C_1^* = C_2 \oplus C_2^*$ such that $B \oplus C \oplus Y = X \oplus Y \oplus fC_1 \oplus B_2^* \oplus C_2^*$, and hence $C_1 \cong fC_1 \cong B_2 \oplus C_2$. Note that $C_1 \cap C_2 = 0$, and so we have that $X \oplus Y \oplus fC_1 \oplus fC_2 \leq \bigoplus B \oplus C \oplus Y = X \oplus Y \oplus fC_1 \oplus B_2^* \oplus C_2^*$. Then there exist decompositions

$B_2^* = B_3 \oplus B_3^*$ and $C_2^* = C_3 \oplus C_3^*$ such that $B \oplus C \oplus Y = X \oplus Y \oplus fC_1 \oplus fC_2 \oplus B_3^* \oplus C_3^*$, and hence $C_2 \cong fC_2 \cong B_3 \oplus C_3$. Continuing the above procedure, we have decompositions $B_n^* = B_{n+1} \oplus B_{n+1}^*$ and $C_n^* = C_{n+1} \oplus C_{n+1}^*$ such that $B \oplus C \oplus Y = X \oplus Y \oplus fC_1 \oplus \cdots \oplus fC_n \oplus B_{n+1}^* \oplus C_{n+1}^*$ and $C_n \cong B_{n+1} \oplus C_{n+1}$. The proof is complete. \square

Theorem 2.2. *Let R be a regular ring with weak comparability, and let A, B and C be finitely generated projective R -modules such that C is directly finite. If $A \oplus C < B \oplus C$, then $A < B$.*

Proof. We may assume with no loss of generality that C is cyclic directly finite projective. To see this notice that, since C is a finitely generated projective R -module, there exists a cyclic decomposition $C = C_1 \oplus \cdots \oplus C_n$ of C . Note that $A \oplus C < B \oplus C$, and so $A \oplus C_1 \oplus \cdots \oplus C_n < B \oplus C_1 \oplus \cdots \oplus C_n$. Since C is directly finite, we see that C_n is cyclic directly finite projective, and so $A \oplus C_1 \oplus \cdots \oplus C_{n-1} < B \oplus C_1 \oplus \cdots \oplus C_{n-1}$ by the assumption. Continuing the above procedure, we have that $A < B$. Since $A \oplus C < B \oplus C$, we have a decomposition $A' \oplus C' \oplus D = B \oplus C$ such that $A' \cong A$, $C' \cong C$ and $D \neq 0$. Let f be an isomorphism from C to C' . Note that $D \oplus C' \leq_{\oplus} B \oplus C$. By Lemma 2.1, there exist decompositions $B = B_1 \oplus B_1^*$ and $B_n^* = B_{n+1} \oplus B_{n+1}^*$ ($n \geq 1$); $C = C_1 \oplus C_1^*$ and $C_n^* = C_{n+1} \oplus C_{n+1}^*$ ($n \geq 1$) such that $D \cong B_1 \oplus C_1$, $C_n \cong B_{n+1} \oplus C_{n+1}$ and $B \oplus C = D \oplus fC_1 \oplus \cdots \oplus fC_n \oplus B_{n+1}^* \oplus C_{n+1}^*$ for each $n = 0, 1, 2, \dots$, where $C_0 = 0$. Notice that $B_k \neq 0$ for some positive integer k , since if $B_k = 0$ for all positive integers k , then we see that $(0 \neq) \aleph_0 D \cong C_1 \oplus C_2 \oplus \cdots \leq C$, which is a contradiction for the direct finiteness of C by Theorem 1.4. Since $A' \oplus C' \oplus D = B \oplus C$, we have that $A' \oplus fC_k^* \oplus (D \oplus fC_1 \oplus \cdots \oplus fC_{k-1}) \leq_{\oplus} A' \oplus fC_1 \oplus \cdots \oplus fC_{k-1} \oplus fC_k \oplus fC_k^* \oplus D = B_k^* \oplus C_k^* \oplus (D \oplus fC_1 \oplus \cdots \oplus fC_{k-1})$. Again by Lemma 2.1, there exist decompositions $B_k^* = B_{k1} \oplus B_{k1}^*$ and $B_{kn}^* = B_{k,n+1} \oplus B_{k,n+1}^*$ ($n \geq 1$); $C_k^* = C_{k1} \oplus C_{k1}^*$ and $C_{kn}^* = C_{k,n+1} \oplus C_{k,n+1}^*$ ($n \geq 1$) such that $A' \cong B_{k1} \oplus C_{k1}$ and $C_{kn} \cong B_{k,n+1} \oplus C_{k,n+1}$ ($n \geq 1$). Then we have that $nC_{kn} \lesssim C \lesssim R$ for each positive integer n . Thus we distinguish to cases.

Case 1. Assume that there exists a positive integer m such that $C_{km} = 0$. Then we have that $A \cong A' \cong B_{k1} \oplus C_{k1} \cong B_{k1} \oplus B_{k2} \oplus C_{k2} \cong \cdots \cong B_{k1} \oplus \cdots \oplus B_{k,m-1} \oplus C_{k,m-1} \cong B_{k1} \oplus \cdots \oplus B_{km} < B_k \oplus B_{k1} \oplus \cdots \oplus B_{km} \leq B$, and so $A < B$ as desired.

Case 2. Assume that $C_{km} \neq 0$ for all positive integers m . Since $B_k \neq 0$, there exists a nonzero cyclic submodule Z of B_k . Thus using the weak comparability for R , we have that $C_{kl} \lesssim Z \leq B_k$ for some positive integer $l = n(Z)$. Notice that $B_{ks} \neq 0$ for some positive integer $s > l$, since if $B_{ks} = 0$ for all positive integers $s > l$, then $(0 \neq) C_{kl} \cong C_{k,l+1} \cong \cdots$ and $C_{kl} \oplus C_{k,l+1} \oplus \cdots \leq C_k^* \leq C$, whence $\aleph_0 C_{kl} \lesssim C$, which is a contradiction for the direct finiteness of C . Therefore $A \cong B_{k1} \oplus C_{k1} \cong B_{k1} \oplus B_{k2} \oplus C_{k2} \cong \cdots \cong B_{k1} \oplus \cdots \oplus B_{kl} \oplus C_{kl} < B_k \oplus B_{k1} \oplus \cdots \oplus B_{kl} \oplus B_{ks} \leq B$. Therefore $A < B$. The proof is complete. \square

From Theorem 2.2 and Corollary 1.6(2), we have the following corollary.

Corollary 2.3. *Let R be a directly finite regular ring with weak comparability, and let A, B and C be finitely generated projective R -modules. If $A \oplus C \prec B \oplus C$, then $A \prec B$.*

Proposition 2.4. *Let R be a regular ring with weak comparability. Let $A \oplus C \oplus D \cong B \oplus C$ for some R -modules A, B, C and D such that C is directly finite finitely generated projective and D has a nonzero finitely generated projective submodule as a direct summand. Then $A \prec_{\oplus} B$.*

Proof. We modify the well-known proof of Fuchs [6, Theorem 2]. Let D^* be a nonzero finitely generated projective submodule of D such that $D^* \leq_{\oplus} D$. Putting that $M = B \oplus C$, we have $M = A_0 \oplus C_0 \oplus D_0$ for some $A_0 \cong A$, $C_0 \cong C$, and $D_0 \cong D$ ($\neq 0$). Since $C \leq_{\oplus} M$ and C has the exchange property, there exists a decomposition $M = C \oplus A' \oplus C' \oplus D'$, where $A_0 = A' \oplus A''$, $C_0 = C' \oplus C''$ and $D_0 = D' \oplus D''$. Then

$$A'' \oplus C'' \oplus D'' \cong C \cong C' \oplus C'' \dots \quad (\sharp)$$

Thus we distinguish to cases.

Case 1. Assume that $D'' \neq 0$. Since C is directly finite finitely generated projective, then $A'' \prec_{\oplus} C'$ by (\sharp) and Theorem 2.2. Hence we have that $A \cong A_0 \prec_{\oplus} A_0 \oplus D' = A' \oplus A'' \oplus D' \prec_{\oplus} A' \oplus C' \oplus D' \cong M/C \cong B$. Therefore $A \prec_{\oplus} B$.

Case 2. Assume that $D'' = 0$, and so $D_0 = D'$. From (\sharp) , we have that $A'' \oplus C'' \cong C' \oplus C'' \prec_{\oplus} C' \oplus C'' \oplus D^*$. Again by Theorem 2.2, we see that $A'' \prec_{\oplus} C' \oplus D^*$, and hence $A \cong A_0 = A' \oplus A'' \prec_{\oplus} A' \oplus C' \oplus D^* \leq_{\oplus} A' \oplus C' \oplus D \cong A' \oplus C' \oplus D' \cong M/C \cong B$. Therefore $A \prec_{\oplus} B$. The proof is complete. \square

Notation. We use $\text{SubP}(R)$ to denote the family of all R -modules M such that $M \lesssim P$ for some projective R -module P .

Lemma 2.5. *Let R be a regular ring, and let $M \in \text{SubP}(R)$. If X is a cyclic R -module with $X \lesssim M$, then X is cyclic projective and $X \lesssim_{\oplus} M$.*

Proof. Since X is cyclic and subisomorphic to M in $\text{SubP}(R)$, we have that X is cyclic projective by [7, Theorem 1.11], and thus $X \lesssim_{\oplus} M$ by the modular law. \square

Theorem 2.6. *Let R be a regular ring with weak comparability. Then for $M \in \text{SubP}(R)$, if M contains no infinite direct sums of nonzero pairwise isomorphic submodules, then so does nM for all positive integers n .*

Proof. We modify the proof of [13, Proposition 2.4]. Let M be an R -module which is in $\text{SubP}(R)$ and contains no infinite direct sums of nonzero pairwise isomorphic submodules. By [7, Corollary 5.6], M is directly finite. Assume that there exists a nonzero cyclic R -module X such that $\aleph_0 X \lesssim 2M$. Then X is cyclic projective and $X \lesssim_{\oplus} 2M$ by Lemma 2.5. Since X has the exchange property, there exist direct summands M_i ($i = 1, 2$)

of M such that $X \cong M_1 \oplus M_2$ and each M_i is directly finite cyclic projective. Take $L = M_1 + M_2$, and notice that $L \leq_{\oplus} M$. If $N = M_1 \cap M_2$, then by [7, Lemma 2.2] we have decompositions $M_1 = M'_1 \oplus N$ and $M_2 = M'_2 \oplus N$. Thus $L = M'_1 \oplus N \oplus M'_2$ and $X \cong (M'_1 \oplus N) \oplus (M'_2 \oplus N) \lesssim_{\oplus} 2L$. Since R has special (DF) by Theorem 1.8 and L is directly finite finitely generated projective, we see that X is directly finite cyclic projective. Since $X \lesssim 2L$, there exists a decomposition $X = A_1 \oplus B_1$ with $A_1 \lesssim B_1 \lesssim L (\leq M)$ by Lemma 1.7. Hence we have that B_1 is directly finite cyclic projective and $X \lesssim_{\oplus} 2B_1$. Since $0 \neq B_1 \lesssim_{\oplus} M$ by Lemma 2.5, we have $M \cong B_1 \oplus D_1$ for some nonzero R -module D_1 , since if $D_1 = 0$ then $M \cong B_1$, and so $2B_1 < 2^2 B_1 \leq 2^2 X \lesssim 2M \cong 2B_1$. But $2B_1$ is directly finite cyclic projective from special (DF) for R , which is a contradiction. For each positive integer $l (\geq 4)$, we have that $lX \oplus E \cong 2M \cong 2B_1 \oplus 2D_1 \leq_{\oplus} 2X \oplus 2D_1$ for some E , and $(l-3)X \oplus 2X \oplus (X \oplus E \oplus F) \cong 2D_1 \oplus 2X$ for some F . Since $2X$ is directly finite finitely generated projective, we have that $(l-3)X <_{\oplus} 2D_1$ by Proposition 2.4. Therefore $tX \lesssim 2D_1$ for all positive integers t . Note that $D_1 \lesssim_{\oplus} M$ by Lemma 2.5 and D_1 contains no infinite direct sums of nonzero pairwise isomorphic submodules. Applying the above argument to D_1 , we have $D_1 \cong B_2 \oplus D_2$ for some nonzero R -modules B_2 and D_2 such that B_2 is directly finite cyclic projective, $X \lesssim_{\oplus} 2B_2$ and $tX \lesssim 2D_2$ for all positive integers t . Continuing the above procedure, we have a projective R -module $B_1 \oplus B_2 \oplus \cdots (\lesssim M)$ such that each B_i is cyclic projective and $X \lesssim_{\oplus} 2B_i$ for all positive integers i . Thus $\aleph_0 X \lesssim_{\oplus} 2(B_1 \oplus B_2 \oplus \cdots)$, and hence $2(B_1 \oplus B_2 \oplus \cdots)$ is directly infinite projective. Since R has special (DF) by Theorem 1.8, we see that $B_1 \oplus B_2 \oplus \cdots$ is directly infinite, and so there exists a nonzero R -module Y such that $\aleph_0 Y \lesssim B_1 \oplus B_2 \oplus \cdots \lesssim M$, which contradicts the assumption. Therefore $2M$ contains no infinite direct sums of nonzero pairwise isomorphic submodules, and so does nM for all positive integers n . The proof is complete. \square

Lemma 2.7. *Let R be a regular ring with weak comparability. Let A, B be finitely generated projective R -modules such that A is directly finite, and let n be a positive integer. Assume that $kC < kD$ implies $C < D$ for every positive integer $k (< n)$ and every finitely generated projective R -modules C and D such that C is directly finite. If $nA < nB$, then there exist decompositions $\overline{\overline{A}}_i = \overline{A}_{i+1} \oplus \overline{\overline{A}}_{i+1}$ and $\overline{\overline{B}}_i = \overline{B}_{i+1} \oplus \overline{\overline{B}}_{i+1}$ such that $\overline{A}_{i+1} \cong \overline{B}_{i+1}$, $2\overline{\overline{A}}_{i+1} \lesssim \overline{\overline{A}}_i$, $\overline{\overline{A}}_{i+1} \lesssim (n-1)\overline{\overline{A}}_{i+1}$, and $n\overline{\overline{A}}_{i+1} < n\overline{\overline{B}}_{i+1}$ for each $i = 0, 1, 2, \dots$, where $\overline{\overline{A}}_0 = A$ and $\overline{\overline{B}}_0 = B$.*

Proof. Let $n = 1$. Since $A < B$, we have a decomposition $B = B_1^* \oplus B_1^{**}$ such that $A \cong B_1^*$ and $B_1^{**} \neq 0$. We put $\overline{A}_1 = A$, $\overline{\overline{A}}_1 = 0$, $\overline{B}_1 = B_1^*$, and $\overline{\overline{B}}_1 = B_1^{**}$; $\overline{A}_i = \overline{\overline{A}}_i = 0$, $\overline{B}_i = 0$, and $\overline{\overline{B}}_i = B_1^{**}$ for each positive integer $i (\geq 2)$ as desired. Let $n (\geq 2)$ be a positive integer. First we claim that there exist decompositions $A = \overline{\overline{A}}_0 = \overline{A}_1 \oplus \overline{\overline{A}}_1$ and $B = \overline{\overline{B}}_0 = \overline{B}_1 \oplus \overline{\overline{B}}_1$ such that $\overline{A}_1 \cong \overline{B}_1$, $2\overline{\overline{A}}_1 \lesssim \overline{\overline{A}}_0$, $\overline{\overline{A}}_1 \lesssim (n-1)\overline{\overline{A}}_1$, and $n\overline{\overline{A}}_1 < n\overline{\overline{B}}_1$. Since $A \lesssim nB$, we have a decomposition $A = A_{11} \oplus \cdots \oplus A_{1n}$ such that $A_{1n} \lesssim \cdots \lesssim A_{11} \lesssim B$ by Lemma 1.7. Setting that $A_1^* = A_{11}$ and $A_1^{**} = A_{12} \oplus \cdots \oplus A_{1n}$, we have $A = A_1^* \oplus A_1^{**}$. Noting that $A_1^* \lesssim_{\oplus} B$, we have a decomposition $B = B_1^* \oplus B_1^{**}$ such that $A_1^* \cong B_1^*$, $A_1^{**} \lesssim (n-1)A_1^*$, and $nA_1^{**} < nB_1^{**}$, because note that $nA < nB$ and nA_1^* is directly finite finitely generated

projective by Theorem 1.8, and so we have $nA_1^{**} < nB_1^{**}$ using Theorem 2.2. Next, since $nA_1^{**} < nB_1^{**}$, we have that $A_1^{**} \lesssim nB_1^{**}$, and hence there exists a decomposition $A_1^{**} = A_{21} \oplus \cdots \oplus A_{2n}$ such that $A_{2n} \lesssim \cdots \lesssim A_{21} \lesssim B_1^{**}$. Setting that $A_2^* = A_{21}$ and $A_2^{**} = A_{22} \oplus \cdots \oplus A_{2n}$, we have that $A_1^{**} = A_2^* \oplus A_2^{**}$ and $A_2^* \lesssim_{\oplus} B_1^{**}$. Then we have a decomposition $B_1^{**} = B_2^* \oplus B_2^{**}$ such that $A_2^* \cong B_2^*$, $A_2^{**} \lesssim (n-1)A_2^*$, and $nA_2^{**} < nB_2^{**}$. Continuing the above procedure $(n-2)$ times, we have decompositions $A_{n-1}^{**} = A_n^* \oplus A_n^{**}$ and $B_{n-1}^{**} = B_n^* \oplus B_n^{**}$ such that $A_n^* \cong B_n^*$, $A_n^{**} \lesssim (n-1)A_n^*$, and $nA_n^{**} < nB_n^{**}$, where $A_n^* = A_{n1}$ and $A_n^{**} = A_{n2} \oplus \cdots \oplus A_{nn}$. Now we put $\bar{A}_1 = A_1^* \oplus \cdots \oplus A_n^*$, $\bar{\bar{A}}_1 = A_n^{**}$, $\bar{B}_1 = B_1^* \oplus \cdots \oplus B_n^*$ and $\bar{\bar{B}}_1 = B_n^{**}$. Then we see that $A = \bar{A}_1 \oplus \bar{\bar{A}}_1$ and $B = \bar{B}_1 \oplus \bar{\bar{B}}_1$ such that $\bar{A}_1 \cong \bar{B}_1$, $\bar{\bar{A}}_1 = A_n^{**} \lesssim (n-1)A_n^* \lesssim (n-1)\bar{A}_1$, and $n\bar{\bar{A}}_1 < n\bar{\bar{B}}_1$. We show that $2\bar{\bar{A}}_1 \lesssim A$. We may assume that $A_n^* \neq 0$, because if $A_n^* = 0$, then $\bar{\bar{A}}_1 = A_n^{**} \lesssim (n-1)A_n^* = 0$ and hence we have that $2\bar{\bar{A}}_1 \lesssim A$, as desired. Note that $(n-1)\bar{\bar{A}}_1 = (n-1)A_n^{**} < A_1^{**} \oplus \cdots \oplus A_{n-1}^{**} \lesssim (n-1)(A_1^* \oplus \cdots \oplus A_{n-1}^*)$. Using the assumption for $k = n-1$, we have $\bar{\bar{A}}_1 < A_1^* \oplus \cdots \oplus A_{n-1}^*$, and so $2\bar{\bar{A}}_1 \cong \bar{\bar{A}}_1 \oplus \bar{\bar{A}}_1 < (A_1^* \oplus \cdots \oplus A_{n-1}^*) \oplus \bar{\bar{A}}_1 < \bar{A}_1 \oplus \bar{\bar{A}}_1 = \bar{A}_0 = A$. Therefore the first claim is proved. Secondly noting that $n\bar{\bar{A}}_1 < n\bar{\bar{B}}_1$, from the above discussion, we have decompositions $\bar{\bar{A}}_1 = \bar{\bar{A}}_2 \oplus \bar{\bar{A}}_2$ and $\bar{\bar{B}}_1 = \bar{\bar{B}}_2 \oplus \bar{\bar{B}}_2$ such that $\bar{\bar{A}}_2 \cong \bar{\bar{B}}_2$, $2\bar{\bar{A}}_2 \lesssim \bar{\bar{A}}_1$, $\bar{\bar{A}}_2 \lesssim (n-1)\bar{\bar{A}}_2$, and $n\bar{\bar{A}}_2 < n\bar{\bar{B}}_2$. Continuing the above procedure, we have desired decompositions. The proof is complete. \square

Lemma 2.8. *Let R be a regular ring with weak comparability. Assume that $nC < nD$ implies $C < D$ for any positive integer n and any finitely generated projective R -modules C and D such that C is cyclic and D is directly finite. Then $nA < nB$ implies $A < B$ for any positive integer n and any finitely generated projective R -modules A and B such that B is directly finite.*

Proof. For any cyclic decomposition $A = A_1 \oplus \cdots \oplus A_m$ of A , $nA < nB$ implies $nA_1 < nB$. Noting that A_1 is cyclic projective, we see that $A_1 < B$ by the assumption, and so there exists a decomposition $B = B_1 \oplus B_1^*$ such that $A_1 \cong B_1$ and $B_1^* \neq 0$. Since $n(A_1 \oplus \cdots \oplus A_m) < n(B_1 \oplus B_1^*)$ and $nA_1 (\cong nB_1)$ is finitely generated directly finite projective by Theorem 1.8, we have that $n(A_2 \oplus \cdots \oplus A_m) < nB_1^*$ by Theorem 2.2. Continuing the above procedure, there exists a decomposition $B_1^* = B_2 \oplus B_2^*$ such that $A_2 \cong B_2$, $B_2^* \neq 0$, and $n(A_3 \oplus \cdots \oplus A_m) < nB_2^*$. Therefore we have that $nA_m < nB_{m-1}^*$, and so $A_m < B_{m-1}^*$ by the assumption. Then we have a decomposition $B_{m-1}^* = B_m \oplus B_m^*$ such that $A_m \cong B_m$ and $B_m^* \neq 0$. Thus $A = A_1 \oplus \cdots \oplus A_m \cong B_1 \oplus \cdots \oplus B_m < B_1 \oplus \cdots \oplus B_{m-1} \oplus (B_m \oplus B_m^*) = B_1 \oplus \cdots \oplus (B_{m-1} \oplus B_{m-1}^*) = \cdots = B_1 \oplus B_1^* = B$, as desired. \square

Lemma 2.9. *Let R be a ring, and let $D = B_1 \oplus \cdots \oplus B_k$ such that $(0 \neq) B_k \lesssim \cdots \lesssim B_1$. If $B_k = E_1 \oplus E_2$ such that $(0 \neq) E_2 \lesssim E_1$, then $D \cong B'_1 \oplus \cdots \oplus B'_{k+1}$ such that $(0 \neq) B'_{k+1} \lesssim \cdots \lesssim B'_1$, where $B'_i = B_i$ ($1 \leq i \leq k-1$), $B'_k = E_1$, and $B'_{k+1} = E_2$.*

Now using Lemmas 2.7–2.9, we show that the strict unperforation property holds for the family of directly finite finitely generated projective modules over regular rings with weak comparability as follows.

Theorem 2.10. *Let R be a regular ring with weak comparability. Then $nA < nB$ implies $A < B$ for any positive integer n and any finitely generated projective R -modules A and B such that B is directly finite.*

Proof. We shall prove the theorem using induction on n . Let $n (\geq 2)$ be a positive integer, and assume that $kA' < kB'$ implies $A' < B'$ for each $k (\leq n - 1)$ and each finitely generated projective R -modules A' and B' such that B' is directly finite. Let A and B be finitely generated projective R -modules such that B is directly finite and $nA < nB$. Then we may assume that A is nonzero cyclic by Lemma 2.8. Since $nA < nB$, there exists a nonzero finitely generated projective R -module D such that $nA \oplus D \cong nB$. By Lemma 2.7 and Theorem 1.8, there exist decompositions $\overline{\overline{A}}_i = \overline{\overline{A}}_{i+1} \oplus \overline{\overline{A}}_{i+1}$ and $\overline{\overline{B}}_i = \overline{\overline{B}}_{i+1} \oplus \overline{\overline{B}}_{i+1}$ such that $\overline{\overline{A}}_{i+1} \cong \overline{\overline{B}}_{i+1}$, $2\overline{\overline{A}}_{i+1} \lesssim \overline{\overline{A}}_i$ for each $i = 0, 1, 2, \dots$, where $\overline{\overline{A}}_0 = A$ and $\overline{\overline{B}}_0 = B$. Notice that $2^{i+1}\overline{\overline{A}}_{i+1} \lesssim \overline{\overline{A}}_0 = A \lesssim R$. If D has a simple submodule S , by the weak comparability of R , we have that $\overline{\overline{A}}_m \lesssim S \leq D$ for some positive integer $m = n(S)$. Noting that $2\overline{\overline{A}}_{m+1} \lesssim \overline{\overline{A}}_m$ and S is simple, we see that $\overline{\overline{A}}_{m+1} = 0$ and so $A = \overline{\overline{A}}_1 \oplus \dots \oplus \overline{\overline{A}}_{m+1} \cong \overline{\overline{B}}_1 \oplus \dots \oplus \overline{\overline{B}}_{m+1} \leq B$. If $A \cong B$, then $nB \cong nA <_{\oplus} nB$, which is a contradiction for the direct finiteness of nB by Theorem 1.8. Thus $A < B$ as desired. Therefore we may assume that $\text{Soc}(D) = 0$. Since $D \lesssim nB$, we have $D \cong B_1 \oplus \dots \oplus B_n$ such that $B_n \leq \dots \leq B_1 \leq B$ by Lemma 1.7.

Step 1. We claim that $B_n \neq 0$. Assume that there exists the largest positive integer $k (< n)$ such that $D \cong B_1 \oplus \dots \oplus B_k$, $B_k \leq \dots \leq B_1 \leq B$, and $B_k \neq 0$. Let $B = B_1 \oplus G_1$, and $B_i = B_{i+1} \oplus C_i$ for $1 \leq i \leq k - 1$. Noting that $\text{Soc}(B_k) = 0$, there exists a decomposition $B_k = E \oplus C_k$ for some nonzero submodules E and C_k of B_k . Then $D \oplus E \cong B_1 \oplus \dots \oplus B_k \oplus E <_{\oplus} B_1 \oplus \dots \oplus B_k \oplus B_k \lesssim (k + 1)B_1 \lesssim nB \cong nA \oplus D$. Since D is directly finite finitely generated projective, we have $E < nA$ by Theorem 2.2. Noting that $E \lesssim nA$ and $B_k = E \oplus C_k$, we see that $E \lesssim A$ by Lemmas 1.7 and 2.9. Therefore there exists a submodule A^1 of A such that $A^1 \cong E$. Let $B = B_k \oplus (B_k)^c$ and $E^c = C_k \oplus (B_k)^c$. Then $B = E \oplus E^c$ and $D \oplus (n - k)A^1 \oplus A^1 \leq D \oplus nA \cong nB \cong nE \oplus nE^c = kE \oplus (n - k)E \oplus nE^c <_{\oplus} D \oplus (n - k)E \oplus nE^c$. Noting that $A^1 \cong E$, we see that $A^1 < nE^c$ by Theorem 2.2, and so $A^1 \lesssim E^c$ by Lemma 2.9. Since $A^1 \lesssim_{\oplus} E^c = G_1 \oplus C_1 \oplus \dots \oplus C_k$ and A^1 has the exchange property, there exist $F \leq G_1$ and $F_i \leq C_i$ ($i = 1, 2, \dots, k$) such that $A^1 \cong F \oplus F_1 \oplus \dots \oplus F_k$. If $F_i \neq 0$ for some i , then we have that $F_i \lesssim A^1 \cong E (\leq B_k)$, $D \cong B_1 \oplus \dots \oplus B_{i-1} \oplus (B_{i+1} \oplus (F_i)^c) \oplus B_{i+1} \oplus \dots \oplus B_k \oplus F_i$ and $(0 \neq) F_i \lesssim B_k \leq \dots \leq B_{i+1} \leq (B_{i+1} \oplus (F_i)^c) \leq B_{i-1} \leq \dots \leq B_1 \leq B$ (where $C_i = F_i \oplus (F_i)^c$ and $B_{k+1} = E$), which is a contradiction for a decomposition of D . Hence $F_i = 0$ for all i , and so $A^1 \cong F \leq G_1$. Therefore we have a submodule B^1 of G_1 such that $B^1 \cong A^1$. Continuing the above procedure, we have independent families $\{A^1, A^2, \dots\}$ of submodules of A and $\{E, B^1, B^2, \dots\}$ of submodules of B such that $E \cong A^i \cong B^i$ for

$i = 1, 2, \dots$. Then $(0 \neq) \aleph_0 E \cong B^1 \oplus B^2 \oplus \dots \leq B$, which is a contradiction for the direct finiteness of B . Therefore $B_n \neq 0$.

Step 2. We claim that $A < B$. By Step 1, we have $D \cong B_1 \oplus \dots \oplus B_n$ such that $B_n \leq \dots \leq B_1 \leq B$ and $B_n \neq 0$. Since $\text{Soc}(B_n) = 0$, there exists a proper nonzero cyclic submodule Z of B_n , and so $B = Z \oplus W$ for some nonzero submodule W of B . Note that $nA \oplus nZ <_{\oplus} nA \oplus D \cong nB \cong nW \oplus nZ$. By Theorem 2.2, we have $nA < nW$. Using Lemma 2.7 with the induction hypothesis, there exist decompositions $\overline{\overline{V}}_i = \overline{\overline{V}}_{i+1} \oplus \overline{\overline{V}}_{i+1}$ and $\overline{\overline{W}}_i = \overline{\overline{W}}_{i+1} \oplus \overline{\overline{W}}_{i+1}$ such that $\overline{\overline{V}}_{i+1} \cong \overline{\overline{W}}_{i+1}$ and $2\overline{\overline{V}}_{i+1} \lesssim \overline{\overline{V}}_i$ for each $i = 0, 1, 2, \dots$, where $\overline{\overline{V}}_0 = A$ and $\overline{\overline{W}}_0 = W$. Since R has weak comparability and $2^{i+1}\overline{\overline{V}}_{i+1} \lesssim \overline{\overline{V}}_0 = A \lesssim R$ for each $i = 0, 1, 2, \dots$, there exists a positive integer $m = n(Z)$ such that $\overline{\overline{V}}_m \lesssim Z$. Hence $A = \overline{\overline{V}}_1 \oplus \dots \oplus \overline{\overline{V}}_m \oplus \overline{\overline{V}}_m \lesssim \overline{\overline{W}}_1 \oplus \dots \oplus \overline{\overline{W}}_m \oplus Z \leq W \oplus Z = B$. Noting that $nA < nB$, we have that $A < B$. The proof is complete. \square

Remark 2.11. Following Theorem 2.10, we obviously have that for a regular ring R with weak comparability, $nA < nB$ implies $A < B$ for any positive integer n and any projective R -modules A and B such that A is finitely generated and B is directly finite.

Corollary 2.12. *Let R be a directly finite regular ring with weak comparability. If $nA < nB$ for some positive integer n and some finitely generated projective R -modules A and B , then $A < B$.*

Note 2.13. In [8], Goodearl gave an example of a simple unit-regular ring R with weak comparability such that $nA \lesssim nB$ does not imply $A \lesssim B$ for some positive integer n and some finitely generated projective R -modules A and B .

Finally, using the definition of weak comparability for a module (see [13, p. 3339]), [13, Lemma 1.4], and the similar proof of one of [13, Proposition 1.3] with Corollary 2.12, we can show that the property of weak comparability for directly finite regular rings is inherited by matrix rings as follows.

Theorem 2.14. *Let R be a directly finite regular ring. Then the following conditions are equivalent:*

- (a) R satisfies weak comparability.
- (b) For each nonzero finitely generated projective R -module P , the endomorphism ring $\text{End}_R(P)$ of P satisfies weak comparability.
- (c) Every ring S which is Morita equivalent to R satisfies weak comparability.
- (d) For all positive integers n , $M_n(R)$ satisfies weak comparability.
- (e) There exists a positive integer n such that $M_n(R)$ satisfies weak comparability.

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